Semestral examination
First semester 2013-14
Algebra III
B.Math.(Hons.) IInd year
Instructor — B.Sury
Maximum marks 100

Q 1. (10 marks) Prove that a Boolean ring must be commutative, and of characteristic 2. Show further that every finitely generated ideal in a Boolean ring, must be principal.

Q 2. (6+6 marks)
(i) Let A be any finite ring. Prove that there exist positive integers $m \neq n$ so that $x^m = x^n$ for all $x \in A$.
(ii) If A is a commutative ring with at most 5 distinct ideals, then every ideal must be principal.

Q 3. (12 marks)
Suppose $d > 2$ is a square-free integer. Prove that $\mathbf{Z}[\sqrt{-d}]$ is not a UFD.

OR

Let A be a commutative ring with unity. Prove that any set of n generators for the A-module A^n must be a basis.
Hint: Prove and use the fact that a surjective A-module homomorphism from A^n to itself must be injective.

Q 4. (12 marks)
Show that the ideal $(X, 4)$ is not the power of a prime ideal in $\mathbf{Z}[X]$.

OR

(6+6 marks)
Let A be a Noetherian ring.
(i) Show that every ideal of A contains a power of its radical.
(ii) $A[[X]]$ is Noetherian.

Q 5. (13 marks)
Let F be any finite field. Prove that there exists $f \in F[X, Y]$ such that

$$\{(x, y) \in F^2 : f(x, y) = 0\} = \{(0, 0)\}.$$

Q 6. (7+6 marks)
(i) Express $43i - 19$ as a product of irreducibles in $\mathbf{Z}[i]$.
(ii) Show that $X^{10} - 6iX^7 + 8X^3 - 1 + 3i$ is irreducible in $\mathbf{Z}[i][X]$.

Q 7. (11 marks)
Prove that $\mathbf{Z} + X\mathbf{Q}[X]$ is not a UFD.

OR

Let A be a commutative ring with unity. Characterize (with proof) all subsets S of A such that the complement of S is a union of prime ideals.

Q 8. (12 marks)
Show that a finitely generated, torsion-free module over a PID is free.

OR

If $f_1, \cdots, f_k \in \mathbf{C}[X_1, \cdots, X_n]$ are such that they have no common zero in $\mathbf{C}^n$, then prove that there exist $g_1, \cdots, g_k$ in $\mathbf{C}[X_1, \cdots, X_n]$ satisfying $\sum_{i=1}^{k} g_i f_i = 1$.

Q 9. (12 marks)
Let $f \in \mathbf{Z}[X]$ be a monic polynomial such that $f = gh$, with $g, h \in \mathbf{Q}[X]$ monic. Prove that $g, h \in \mathbf{Z}[X]$.

OR

Define the rational canonical form of a matrix over a field. Describe the minimal polynomial and the characteristic polynomial in terms of the invariant factors and prove that they have the same irreducible factors.